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LETTER TO THE EDITOR

On the relevance of orthogonal ($d \times d$) matrices within Nikolskii's transformation method for ($d > 1$)-dimensional Boltzmann equations

Günter Dukek

Department of Mathematical Physics, University of Ulm, D-7900 Ulm, Federal Republic of Germany

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Abstract. The method of Nikolskii is studied for d-dimensional nonlinear Boltzmann equations with momentum and energy conservations holding. Outside forces are taken into account, and a class of cross sections obeying a homogeneity condition is assumed. Cornille's ansatz for the Nikolskii transform method is generalised in accordance with the microscopic conservation laws. The new ansatz, incorporating a space- and time-dependent orthogonal ($d \times d$) matrix, proves to be consistent if the outside force satisfies a compatibility condition. The significance of orthogonal matrices depending only on the time coordinate, is pointed out for linear spatial forces in the context of the average flow velocity.

In the Nikolskii [1–4] transform method (NM) one seeks solutions of the nonlinear Boltzmann equation (BE)

$$Lf = J(f, f) \qquad L = \partial_t + \boldsymbol{v} \cdot \partial_x + \boldsymbol{A}(\boldsymbol{v}, \boldsymbol{x}, t) \cdot \partial_v \tag{1}$$

in the form

$$f(\boldsymbol{v}, \boldsymbol{x}, t) \equiv F(\boldsymbol{V}(\boldsymbol{v}, \boldsymbol{x}, t), \tau(t)) \tag{2}$$

where the functions

$$\boldsymbol{V} = \boldsymbol{V}(\boldsymbol{v}, \boldsymbol{x}, t) \qquad \tau = \tau(t) \tag{3a, b}$$

depending on the time t, space coordinate $\boldsymbol{x} \in \mathbb{R}^d$ and velocity $\boldsymbol{v} \in \mathbb{R}^d$, have to be determined such that (1) reduces to the spatially uniform equation

$$L_0 F = J(F, F) \qquad L_0 = \partial_\tau. \tag{4}$$

For a given type of collision integral $J(f, f)$, it is important to know the admissible transformations (3) and solutions $F(\boldsymbol{V}, \tau)$ of (4). Then, according to (2), one can build up classes of inhomogeneous distributions from homogeneous ones. Usually, the transformations ($3a$) are chosen in close analogy to the ansatz, $\boldsymbol{V} = \gamma \boldsymbol{c}$, known from the study of locally Maxwellian distributions with peculiar velocity $\boldsymbol{c} = \boldsymbol{v} - \boldsymbol{u}$. Such transformations (with γ, $\boldsymbol{u}$ being functions of $\boldsymbol{x}$ and t) prove to be sufficient for energy dependent distributions $F(V^2, \tau)$. However, no solution of the BE (4) need be of that particular type, and then the above assumption appears too restrictive. In this letter we aim at presenting an extended class of transformations ($3a$) for the ($d > 1$)-dimensional BE (1) with

$$J(f, f) = S_d^{-1} \int \mathrm{d}\Omega_d \, \mathrm{d}\boldsymbol{w} |\boldsymbol{v} - \boldsymbol{w}| \sigma(|\boldsymbol{v} - \boldsymbol{w}|, \cos\chi)[f(\boldsymbol{v}')f(\boldsymbol{w}') - f(\boldsymbol{v})f(\boldsymbol{w})] \tag{5}$$

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with $S_d = \int \mathrm{d}\Omega_d$, momentum and energy conservation holding:

$$\boldsymbol{\delta} = \boldsymbol{v} + \boldsymbol{w} - (\boldsymbol{v}' + \boldsymbol{w}') = \mathbf{0} \qquad \delta_0 = v^2 + w^2 - (v'^2 + w'^2) = 0 \tag{6}$$

and the cross section σ being a homogeneous function of degree e with respect to the first argument:

$$\sigma(\lambda |\boldsymbol{v} - \boldsymbol{w}|, \cos\chi) = \lambda^e \sigma(|\boldsymbol{v} - \boldsymbol{w}|, \cos\chi). \tag{7}$$

The molecular interactions characterised by such a functional relationship (λ arbitrary) contain the intermolecular forces with inverse power law p [3] as a special case, namely for $e = -2(d-1)/(p-1)$. The formulae expressing the velocities after impact ($\boldsymbol{v}'$, $\boldsymbol{w}'$) as functions of angle variables χ, θ, ε and velocities before impact ($\boldsymbol{v}$, $\boldsymbol{w}$), as well as the details of the d-dimensional solid angle integration may be found elsewhere [3]. In the spatially homogeneous BE (4), it is understood that $J(F, F)$ is the expression written down in (5) with $\boldsymbol{v}'$, $\boldsymbol{w}', \ldots$ replaced by $\boldsymbol{V}$, $\boldsymbol{W}, \ldots$ always with

$$\boldsymbol{\Delta} = \boldsymbol{V} + \boldsymbol{W} - (\boldsymbol{V}' + \boldsymbol{W}') = \mathbf{0} \qquad \Delta_0 = V^2 + W^2 - (V'^2 + W'^2) = 0. \tag{8}$$

Cornille's ansatz for the NM is $\boldsymbol{V} = \gamma(\boldsymbol{x}, t)(\boldsymbol{v} - \boldsymbol{v}_0(\boldsymbol{x}, t))$. Clearly, for γ non-zero, these linear transformations between the velocities $\boldsymbol{V}$ and $\boldsymbol{v}$ constitute a group admitting $M_0(d) = 1 + d$ (continuous and differentiable) functions of $\boldsymbol{x}$ and t. This group is assumed in the NM even if the BE is markedly different from (1) as, for instance, the BE of a gas of test particles interacting with a background host medium [4, 5]. Here, we propose an extended transformation group:

$$\boldsymbol{V} = \gamma(\boldsymbol{x}, t)\mathbf{B}(\boldsymbol{x}, t)(\boldsymbol{v} - \boldsymbol{v}_0(\boldsymbol{x}, t)) \tag{9a}$$

$$\tilde{\mathbf{B}}(\boldsymbol{x}, t)\mathbf{B}(\boldsymbol{x}, t) = \mathbf{E} \tag{9b}$$

with $\gamma(\boldsymbol{x}, t) \neq 0$, $\mathbf{E}$ being the unit matrix, $\mathbf{B}(\boldsymbol{x}, t)$ an orthogonal $(d \times d)$ matrix, and $\tilde{\mathbf{B}}(\boldsymbol{x}, t)$ its transpose. For that group, admitting $M(d) = M_0(d) + \frac{1}{2}d(d-1)$ space- and time-dependent functions, one can show that (6) and (8) follow from each other. Indeed, the simultaneous transformations $\boldsymbol{V} = \gamma\mathbf{B}(\boldsymbol{v} - \boldsymbol{v}_0)$, $\boldsymbol{W} = \gamma\mathbf{B}(\boldsymbol{w} - \boldsymbol{v}_0), \ldots$ induce a non-degenerate linear transformation between $\{\boldsymbol{\Delta}, \Delta_0\}$ and $\{\boldsymbol{\delta}, \delta_0\}$

$$\boldsymbol{\Delta} = \gamma\mathbf{B}\boldsymbol{\delta} \qquad \Delta_0 = \gamma^2\{\delta_0 - 2\boldsymbol{\delta} \cdot \boldsymbol{v}_0\} \tag{10}$$

having the property $\{\boldsymbol{\Delta} = \mathbf{0}, \Delta_0 = 0\} \Leftrightarrow \{\boldsymbol{\delta} = \mathbf{0}, \delta_0 = 0\}$. Hence, the extended transformation group is in accordance with the requirements of the microscopic conservation laws. This observation indicates that (9) is the adequate basis for the NM. Cornille's ansatz corresponds to the trivial solution of (9b), $\mathbf{B}(\boldsymbol{x}, t) \equiv \mathbf{E}$. In the NM based on (9) one may choose $\mathbf{B}(\boldsymbol{x}, t) \equiv \mathbf{E}$ without loss of generality in any equation depending on V^2. This justifies Cornille's assumption for energy-dependent distributions $F(V^2, \tau)$, such as the BKW even velocity mode [2, 6], but not for other types of distributions F. According to this rule it is also clear that the extended group (9) does not enlarge the class of inhomogeneous similarity solutions [2, 3] of the BE, being generated by the BKW mode. In that respect it would be desirable to know other distributions F, not depending on V^2. Unfortunately, at present, only for Maxwell particles (i.e. $e = -1$ in (7)) and even velocity distributions do we completely know the method for the explicit construction of distributions F.

We intend to discuss now some consequences of (9) within the NM on both the microscopic and macroscopic level of description. We address those problems which can be treated without explicit knowledge of $F(\boldsymbol{V}, \tau)$.

A direct consequence of (9) is

$$\frac{(V-W)\cdot(V'-W')}{|V-W||V'-W'|}=\frac{(v-w)\cdot(v'-w')}{|v-w||v'-w'|}:=\cos\chi. \tag{11}$$

Equation (11), expressing the conservation of the centre-of-mass scattering angle χ under the above transformations, justifies that the same angle variable $\chi\in[0,\pi]$ can be used within both collision integrals $J(F,F)$ and $J(f,f)$. The following conditions ensure that the inhomogeneous BE can be reduced to the homogeneous one:

$$\dot{\tau}J(F,F)=J(f,f) \qquad \dot{\tau}L_0F=Lf \tag{12a, b}$$

(a superimposed dot indicates the usual time derivative). Observing the relation between the collision terms

$$J(F,F)=|\det \mathbf{B}(x,t)||\gamma(x,t)|^{1+d+e}J(f,f) \tag{13}$$

(following from the definition of $J(F,F)$ together with (5), (7), (9), (11)) and evaluating Lf by means of (2), one obtains from (12)

$$\dot{\tau}(t)|\det \mathbf{B}(x,t)||\gamma(x,t)|^{1+d+e}=1 \qquad (\partial_V F)\cdot(LV)=0. \tag{14a, b}$$

The determining equation for $\tau(t)$, equation (14a), requires γ to be space independent, but does not imply restrictions upon the orthogonal matrices, since $\det \mathbf{B}(x,t)=\pm 1$. Hence, for $\gamma(x,t)=\gamma_0(t)$, the admissible transformations (3b) are

$$\tau(t)=\int_0^t|\gamma_0(t')|^{-(1+d+e)}\,\mathrm{d}t'. \tag{15}$$

A particular solution of (14b), not depending on the type of distribution $F(V,\tau)$, is $LV=\mathbf{0}$. That condition can be replaced by $LV^2=0$ if the distributions F depend only on the energy. A discussion of the differential equation $LV^2=0$ may be found elsewhere [3]. We concentrate on $LV=\mathbf{0}$ which can be rewritten as

$$A(v,x,t)=-\frac{1}{\gamma_0(t)}\tilde{\mathbf{B}}(x,t)\frac{\mathrm{D}}{\mathrm{D}t}\{\gamma_0(t)\mathbf{B}(x,t)[v-v_0(x,t)]\} \tag{16}$$

where $\mathrm{D}/\mathrm{D}t=\partial_t+v\cdot\partial_x$.

Equation (16) can be viewed in two different ways:

(i) for a given choice of γ_0, $\mathbf{B}$, v_0, it defines the compatible outside force A;

(ii) for A being a prescribed function, it represents a partial differential equation (nonlinear in $\mathbf{B}$) for the unknown quantities γ_0, $\mathbf{B}$, v_0.

It is instructive to discuss (16) in connection with the relations between the homogeneous and inhomogeneous macroscopic quantities (number density and mean velocity of molecules, stress tensor and energy flux vector). Here, we confine our considerations to the relation between $U:=\langle V\rangle$ and $u:=\langle v\rangle=\int vf\,\mathrm{d}v/\int f\,\mathrm{d}v$, obtained (formally) from (9a) upon replacing v, V by u, U. We bring this relation into the form

$$u(x,t)=v_0(x,t)+\frac{1}{\gamma_0(t)}\tilde{\mathbf{B}}(x,t)U \tag{17}$$

with $U=\int VF\,\mathrm{d}V/\int F\,\mathrm{d}V$ being constant as a consequence of momentum conservation. When $A=\mathbf{0}$, equation (16) admits the solution

$$\gamma(x,t)=t \qquad \mathbf{B}(x,t)=\mathbf{B}_1 \qquad v_0(x,t)=x/t \tag{18}$$

with $\mathbf{B}_1$ a constant $(d \times d)$ matrix. Here, the integration constants are assumed such that the original Nikolskii variable, $V = t(v - x/t)$, is obtained for the trivial choice $\mathbf{B}_1 = \mathbf{E}$. When inserting (18) into (17), it follows that the possible motions are a superposition of radial and translational motion. The translational motion disappears for even velocity distributions $F(V^2, \tau)$. For this particular class of distributions, Cornille proved that rotational motion can occur in the presence of outside forces depending linearly on x. His proof [3] is based on the assumption $\mathbf{B}(x, t) \equiv \mathbf{E}$ and relies on the differential equation $LV^2 = 0$. In the context of $LV = \mathbf{0}$, the proof is as follows. We write down a generalisation of (18)

$$\gamma(x, t) = \gamma_0(t) \qquad \mathbf{B}(x, t) = \mathbf{B}_0(t) \qquad v_0(x, t) = v_1(t) + \mathbf{M}(t)x. \tag{19}$$

The contributions on the RHS of (16) which are bilinear in v, vanish for any space independent orthogonal matrix $\mathbf{B}_0(t)$. The remaining velocity dependent terms (linear in v) vanish for the particular choice

$$\mathbf{M}_0 = \gamma_0^{-1}\dot{\gamma}_0\mathbf{E} + \boldsymbol{\Omega} \qquad \boldsymbol{\Omega} := \tilde{\mathbf{B}}_0\dot{\mathbf{B}}_0 \tag{20}$$

with $\boldsymbol{\Omega}$ representing a time dependent antisymmetric matrix $\tilde{\boldsymbol{\Omega}} + \boldsymbol{\Omega} = \mathbf{0}$. Under the additional condition (20) one finds the relations

$$A_0(x, t) = [\partial_t + \mathbf{M}_0(t)]v_0(x, t) = [\partial_t + (v_0(x, t) \cdot \partial_x)]v_0(x, t) \tag{21}$$

$$u = v_1 + \gamma_0^{-1}\tilde{\mathbf{B}}_0 U + \gamma_0^{-1}\dot{\gamma}_0 x + \boldsymbol{\Omega}x. \tag{22}$$

In contrast to the derivation in [3], these results hold for any solution $F(V, \tau)$ of the BE (4). In $d = 3$ dimensions the last equation admits an elementary interpretation: Recalling $\boldsymbol{\Omega}x = \omega \times x$, it follows from (22) that the possible motions are a superposition of translational motion, radial motion and rotation with vector angular velocity ω (solid body rotation of the gas).

Finally, by virtue of $\dot{\mathbf{B}}_0 + \boldsymbol{\Omega}\tilde{\mathbf{B}}_0 \equiv \mathbf{0}$, one is allowed to substitute in the last term of (21): $v_0 \rightarrow u$. In that form (21) agrees with the (reduced) Euler equation, obtained from the BE by the method of summational invariants.

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